Decision making with Conditional Value-at-Risk and spectral risk measures, and the problem of comparative risk aversion

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We analyze spectral risk measures with respect to comparative risk aversion following ARROW (1965) and PRATT (1964) on the one hand, and ROSS (1981) on the other hand, and study the implications for the willingness to pay for insurance and portfolio selection. Within the framework of Arrow and Pratt, we show that the widely-applied spectral Arrow-Pratt measure is not a consistent measure of Arrow-Pratt-risk aversion. If still being applied, a decision maker with a greater spectral Arrow-Pratt measure may only be willing to pay less for insurance or to invest more in the risky asset than a decision maker with a smaller spectral Arrow-Pratt measure. Within the framework of Ross, we show that the popular subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures cannot be completely ordered with respect to Ross-risk aversion. As a consequence, these subclasses also exhibit counter-intuitive comparative static results, both in the insurance problem and the portfolio selection problem. In the insurance problem, the willingness to pay for insurance may be decreasing with increasing risk aversion. Likewise, in the portfolio selection problem, the investment in the risky asset may be increasing with increasing risk aversion. Decision makers and regulators should be aware of these shortcomings and its economic consequences before applying spectral risk measures.

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1. Introduction

The theory of risk measurement has faced a paradigm shift in recent times. Following an axiomatic – rather than an ad hoc – approach, ARTZNER ET AL. (1999) have proposed a set of properties that any so-called coherent risk measure in the field of bank regulation and solvency capital requirements should satisfy. Among them, the property of subadditivity has gained particular relevance, as it ensures that issues of diversification are adequately taken into account.

In the course, Conditional Value-at-Risk has been introduced as the most important representative of coherent risk measures (e.g., ACERBI/TASCHE (2002), ROCKAFEL-LAR/URYASEV (2002)), and has been canonically extended towards spectral risk measures (ACERBI (2002)). Since then, spectral risk measures have emerged as the most important subclass of coherent risk measures. Spectral risk measures are characterized by a so-called risk spectrum that assigns subjective weights to the quantiles of a profit and loss (P&L) distribution in order to represent subjective risk aversion. To this end, DOWD ET AL. (2008) have derived the specific subclasses of exponential and power spectral risk measures.

In the recent literature, spectral risk measures and their specific subclasses have also been applied to numerous fields that go beyond the determination of solvency capital requirements, such as portfolio selection (e.g., ADAM ET AL. (2008), BRANDTNER (2013)), the analysis of optimal (re-)insurance contracts (e.g., CAI ET AL. (2008)), and problems in operations management (e.g., JAMMERNEGG/KISCHKA (2007)), to name but a few. These recent applications extend the scope of spectral risk measures from pure regulation-based tools towards models of decision making. Hence, the analysis of spectral risk measures from the perspective of decision theory, addressing standard questions such as the measurement and the comparative statics of risk aversion has become an increasingly relevant issue.

Despite this extended scope of spectral risk measures as means of economic decision making, questions of consistent comparative static results with respect to a decision maker's risk aversion received only little attention so far. That question, however, should have been thoroughly addressed in advance, as it is well-known that even classical expected utility theory may fail in providing consistent comparative static results in standard economic problems such as the willingness to pay for insurance or portfolio selection (e.g., KIHLSTROM ET AL. (1981), ROSS (1981)).

In this paper, we aim to fill this research gap by analyzing spectral risk measures and their popular subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures with respect to comparative risk aversion. Our contribution is twofold:

First, we argue within the classical ARROW (1965) and PRATT (1964) (AP)-framework of (comparative) risk aversion, which has dominated the discussion of risk aversion in hundreds of published papers of economic decision making in the expected utility framework.

Inspired by this classical framework, the so-called spectral Arrow-Pratt measure is regularly employed as a measure of AP-risk aversion in the literature on the new framework of spectral risk measures. In this paper, we show that, contrary to intuition, the spectral Arrow-Pratt measure is not a consistent measure of AP-risk aversion and provide the right measure.

In terms of economic application, we demonstrate that the two standard problems of the willingness to pay for insurance and the portfolio selection between a risk free and a risky asset are covered by the framework of AP-risk aversion. Accordingly, applying the spectral Arrow-Pratt measure to these problems as is done in the literature may yield misleading results: A decision maker with a greater spectral Arrow-Pratt measure may only be willing to pay less for insurance or to invest more in the risky asset than a decision maker with a smaller spectral Arrow-Pratt measure.

Second, we argue within the extended framework of ROSS (1981) (R), who has offered another concept of (comparative) risk aversion for a more general situation where the initial wealth is random and the risk of the final wealth can only be eliminated partly. Accordingly, this framework addresses the shift from "more" to "less" risk, which appears to be the more realistic setting, in particular from the point of view of the recent financial crises. We show that neither Conditional Value-at-Risk, nor exponential and power spectral risk measures can be completely ordered with respect to R-risk aversion. We further provide a general "destructive" result of non-consistency between spectral risk measures and R-risk aversion.

As a consequence, these subclasses of spectral risk measures also exhibit counter-intuitive comparative static results with respect to the respective risk aversion parameters, both in the insurance problem and the portfolio selection problem: In the insurance problem, the willingness to pay for insurance may be decreasing with increasing risk aversion. Likewise, in the portfolio selection problem, the investment in the risky asset may be increasing with increasing risk aversion. Decision makers and regulators should be aware of these shortcomings and its economic consequences before applying spectral risk measures.

The paper proceeds as follows. Section 2 introduces spectral risk measures and the specific subclasses under consideration. Section 3 addresses the concept of risk aversion following ARROW (1965) and PRATT (1964), while Section 4 addresses the concept of ROSS (1981). Section 5 concludes.

2. Setting: Spectral risk measures

Let $\mathcal{X}$ denote the set of all real valued (P&L) random variables X on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $F_X(x) = F(x) = P(X \leq x)$ be the cumulative distribution function of X with corresponding quantile function $F_X^{-1}(p) = F^{-1}(p) = \sup\{x \in \mathbb{R} | F(x) < p\}, p \in (0, 1]$ and $F^{-1}(0) = \lim_{t \to 0^+} F^{-1}(t)$.

2.1. Properties and representation

Spectral risk measures originally have been introduced for determining solvency capital requirements in bank regulation. In order to satisfy this purpose adequately, they have to fulfill a set of properties (or axioms) (ACERBI (2002), ACERBI (2004), Proposition 3.26).

Definition 2.1. *A mapping $\rho_\phi : \mathcal{X} \to \mathbb{R}$ is called spectral risk measure if it satisfies the following properties for all $X, Y, Z \in \mathcal{X}$:*

- *Monotonicity: $X \leq Y \Rightarrow \rho_\phi(X) \geq \rho_\phi(Y)$.*

- *Translation invariance: $\rho_\phi(X + c) = \rho_\phi(X) - c, c \in \mathbb{R}$.*

- *Subadditivity: $\rho_\phi(X + Y) \leq \rho_\phi(X) + \rho_\phi(Y)$.*

- *Positive homogeneity: $\rho_\phi(\lambda \cdot X) = \lambda \cdot \rho_\phi(X), \lambda \geq 0$.*

- *Comonotonic additivity: X, Y comonotonic (i.e., $X = g(Z)$ and $Y = h(Z)$ where g, h non-increasing) $\Rightarrow \rho_\phi(X + Y) = \rho_\phi(X) + \rho(Y)$.*

- *Law invariance: $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R} \Rightarrow \rho_\phi(X) = \rho_\phi(Y)$.*

For a thorough discussion of these properties see ACERBI (2004). In the literature, the property of subadditivity has gained particular attention, as it ensures that spectral risk measures adequately reflect effects of diversification. Besides subadditivity, for our analyses the linearity of spectral risk measures induced by the properties of translation invariance and positive homogeneity

$$\rho_\phi(\lambda \cdot X + c) = \lambda \cdot \rho_\phi(X) + c, c \in \mathbb{R}, \lambda \geq 0 \tag{1}$$

will become relevant.

Spectral risk measures have the following representation.

Theorem 2.2. *Any spectral risk measure $\rho_\phi : \mathcal{X} \to \mathbb{R}$ is of the form*

$$\rho_\phi(X) = -\int_0^1 F^{-1}(p) \cdot \phi(p) \mathrm{d}p, \tag{2}$$

where the so-called risk spectrum $\phi : [0, 1] \to \mathbb{R}_+$ is a non-increasing density function.

For the proof, see ACERBI (2004), Proposition 3.4. Spectral risk measures through the risk spectrum ϕ assign subjective weights to the p-quantiles of a random variable X with

smaller quantiles receiving relatively greater weights and vice versa. The antiderivative of the risk spectrum $\phi(p)$ given by

$$\Phi(p) = \int\limits_0^p \phi(t)\mathrm{d}t \tag{3}$$

is a concave cumulative distribution function on $[0,1]$ with $\Phi(0) = 0$, $\Phi(1) = 1$.

In the recent literature, spectral risk measures have also been applied to numerous fields beyond the determination of solvency capital requirements such as portfolio selection (e.g., ADAM ET AL. (2008), BRANDTNER (2013)), optimal (re-)insurance contracts (e.g., CAI ET AL. (2008)), and problems in operations management (e.g., JAMMERNEGG/KISCHKA (2007)), to name but a few.

These recent applications extend the scope of spectral risk measures from pure regulation-based tools of risk towards models of economic decision making that are used to find optimal solutions over a set of alternatives. In order to capture this new scope, we make use of the following notion.

Definition 2.3. *If a decision maker decides according to*

$$X \text{ is preferred to } Y \Leftrightarrow \rho_\phi(X) \le \rho_\phi(Y), \tag{4}$$

he is called a spectral risk measure (SRM)-decision maker.

Within the extended scope of spectral risk measures as a means of modeling optimal decisions of SRM-decision makers, the tradeoff between risk and reward becomes an important issue. Spectral risk measures regularly represent an implicit risk-reward tradeoff (e.g., ACERBI/SIMONETTI (2002), p. 10). In order to make this tradeoff explicit, assume that the risk spectrum satisfies $\phi(1) > 0$, which is common in the literature (e.g., DOWD ET AL. (2008)), and it is also satisfied for the specific subclasses of spectral risk measures that will be introduced below. Then the risk spectrum can be decomposed by

$$\phi(p) = \phi(1) \cdot 1 + (1 - \phi(1)) \cdot \hat{\phi}(p), \text{ where } \hat{\phi}(p) = \frac{\phi(p) - \phi(1)}{1 - \phi(1)}, \tag{5}$$

such that the corresponding spectral risk measure can be rewritten as

$$\rho_\phi(X) = -(\phi(1) \cdot E(X) - (1 - \phi(1)) \cdot \rho_{\hat{\phi}}(X)). \tag{6}$$

Thus, spectral risk measures can be denoted as (negative) linear combination of the expectation as a reward measure and $\rho_{\hat{\phi}}$ as another spectral measure that captures the "pure" risk.

4

Remark 2.4. Note that a close relationship prevails between spectral risk measures and YAARI (1987)'s dual theory of choice (see also ROELL (1987) and DENNEBERG (1988)). The dual theory requires the existence of some dual utility function $v : [0, 1] \to [0, 1], v(0) = 0, v(1) = 1$ such that the decision maker's preference over a set of risky positions is measured by

$$D_v(X) = \int\limits_0^1 F^{-1}(p)\mathrm{d}v(p). \tag{7}$$

If a decision maker decides according to "X is preferred to $Y \Leftrightarrow D_v(X) \geq D_v(Y)$", he is called a dual theory of choice (DT)-decision maker. The representations of spectral risk measures (2) and the dual theory of choice (7) up to the algebraic sign coincide if one identifies the antiderivative of the risk spectrum Φ with the dual utility function v, with the exception that the dual theory of choice is more general in that v not necessarily needs to be concave. This close relationship allows us to draw on previous results on comparative risk aversion derived for the dual theory of choice below.

2.2. Examples

The most popular spectral risk measure is Conditional Value-at-Risk (e.g., ACERBI/TASCHE (2002), ROCKAFELLAR/URYASEV (2002)). At the confidence level $\alpha \in [0, 1]$, its risk spectrum and its antiderivative are given by

$$\phi_\alpha(p) = \begin{cases} \frac{1}{\alpha} & \text{for } p \in [0, \alpha) \\ 0 & \text{for } p \in [\alpha, 1] \end{cases} \text{ and } \Phi_\alpha(p) = \begin{cases} \frac{1}{\alpha} \cdot p & \text{for } p \in [0, \alpha) \\ 1 & \text{for } p \in [\alpha, 1] \end{cases} \tag{8}$$

(see Figure 1). Conditional Value-at-Risk assigns a constant weight of $\frac{1}{\alpha}$ to the $\alpha \cdot 100\%$ smallest outcomes, while the greater outcomes are not taken into account. As Conditional Value-at-Risk does not assign a positive weight of $\phi_\alpha(1) > 0$ to the $p = 1$-quantile, it initially does not represent a risk-reward tradeoff along the decomposition (6). However, Conditional Value-at-Risk can be extended towards a (spectral) risk-reward tradeoff by forming a negative convex combination of the random variable's mean and Conditional Value-at-Risk, viz

$$\rho_\phi(X) = -(\lambda \cdot E(X) - (1 - \lambda) \cdot CVaR_\alpha(X)); \tag{9}$$

see ACERBI/SIMONETTI (2002) or JAMMERNEGG/KISCHKA (2007) in more detail.

As an alternative to popular Conditional Value-at-Risk, DOWD ET AL. (2008) have proposed exponential spectral risk measures, which in the course have become a relevant subclass of its own (e.g., BARBI/ROMAGNOLI (2013), COTTER/DOWD (2010), DOWD/BLAKE
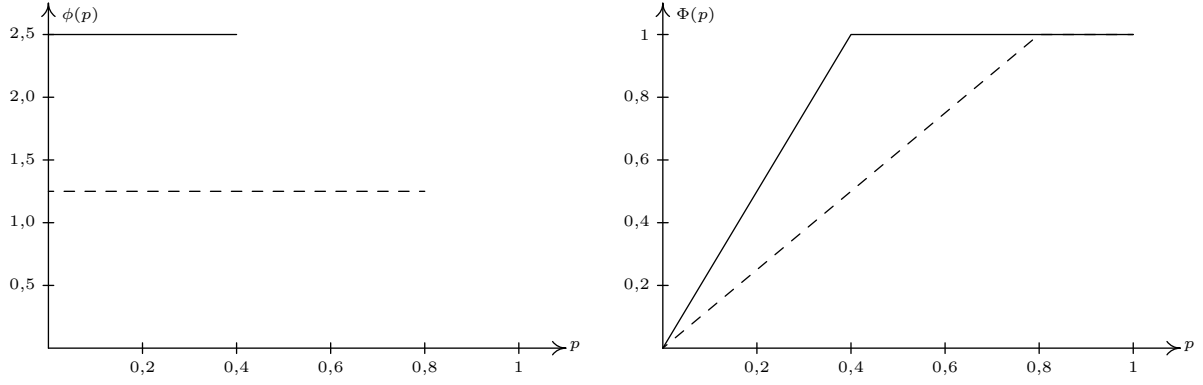
Figure 1: Conditional Value-at-Risk for $\alpha = 0,4$ (solid) and $\alpha = 0,8$ (dashed)
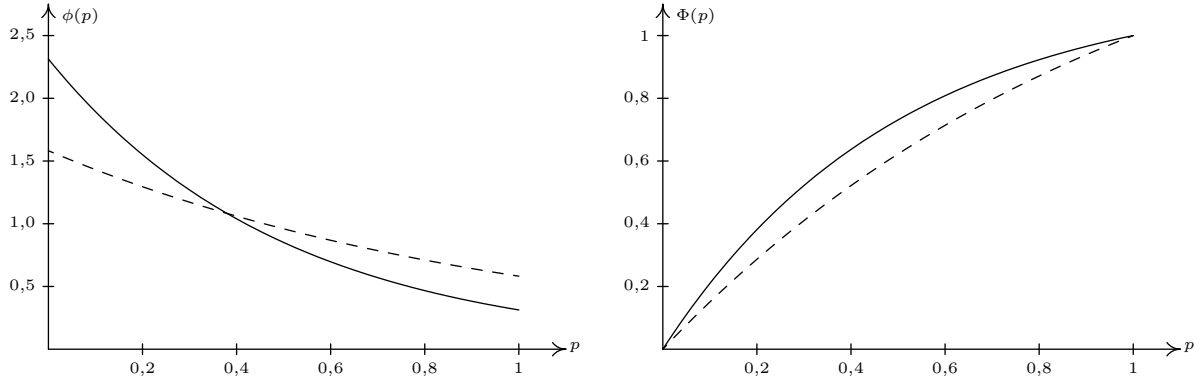

Figure 2: Exponential spectral risk measures for $a = 1$ (dashed) and $a = 2$ (solid)

(2006)). For exponential spectral risk measures, the risk spectrum and its antiderivative for $a > 0$ are given by

$$\phi_a(p) = \frac{a \cdot e^{-a \cdot p}}{1 - e^{-a}}, p \in [0, 1] \text{ and } \Phi_a(p) = \frac{1 - e^{-a \cdot p}}{1 - e^{-a}}, p \in [0, 1] \tag{10}$$

(see Figure 2). As $\phi_a(1) > 0$, exponential spectral risk measures represent an implicitly risk-reward tradeoff as given by (6).

Moreover, DOWD ET AL. (2008) have introduced power spectral risk measures. Here, the risk spectrum and its antiderivative for $0 < b \leq 1$ are given by

$$\phi_b(p) = b \cdot p^{b-1}, p \in [0, 1] \text{ and } \Phi_b(p) = p^b, p \in [0, 1] \tag{11}$$

(see Figure 3). Again it holds that $\phi_b(1) = b > 0$, so for power spectral risk measures the risk-reward tradeoff (6) can be easily made explicit by

$$\rho_{\phi_b}(X) = -(b \cdot E(X) - (1 - b) \cdot \rho_{\hat{\phi}_b}(X)) \text{ with } \hat{\phi}_b(X) = \frac{b \cdot (p^{b-1} - 1)}{1 - b}. \tag{12}$$
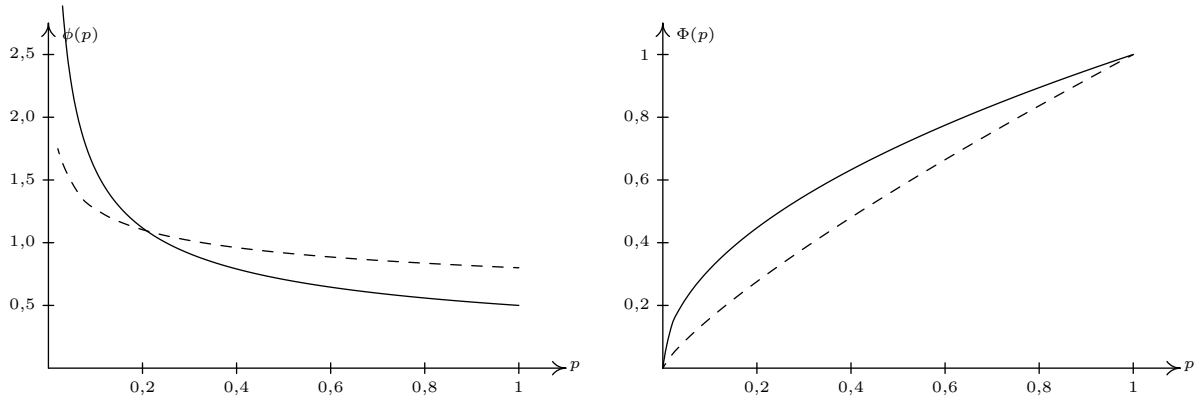
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Figure 3: Power spectral risk measures for $b = 0{,}5$ (solid) and $b = 0{,}8$ (dashed)

3. Comparative AP-risk aversion: The case of deterministic initial wealth

3.1. Definitions

We first address comparative risk aversion following ARROW (1965) and PRATT (1964). Their framework is based on the idea of completely eliminating risk, i.e., it is assumed that a decision maker can switch from a risky position to a risk free position. In order to model this shift, the notion of the certainly equivalent is employed, and is related to the concepts of risk aversion and comparative risk aversion as follows:

Definition 3.1.

a) *The certainty equivalent of a position X, $c(X)$, indicates the certain position for which the decision maker is indifferent to the position X.*

b) *A decision maker is said to be AP-risk averse if $c(X) \leq E(X)$ for all $X \in \mathcal{X}$.*

c) *A decision maker 1 is said to be more AP-risk averse than a decision maker 2 if $c_1(X) \leq c_2(X)(\leq E(X))$ for all $X \in \mathcal{X}$.*

An AP-risk averse decision maker for any risky position X is willing to accept a certain position that is less than the risky position's expectation $E(X)$ to avoid the risk induced by X. The more AP-risk averse decision maker for any risky position X is willing to accept a smaller certain position to avoid X than the less AP-risk averse decision maker.

3.2. Comparative AP-risk aversion for spectral risk measures

We now address the measurement of AP-risk aversion for spectral risk measures. To this end, we first specify the certainty equivalent.

Theorem 3.2. *The certainty equivalent of a SRM-decision maker with risk spectrum ϕ, $c_\phi(X)$, is given by*

$$c_\phi(X) = -\rho_\phi(X). \tag{13}$$

The proof is straightforward: The certainty equivalent is defined by $\rho_\phi(c_\phi(X)) = \rho_\phi(X)$, and translation invariance yields $\rho_\phi(c_\phi(X)) = -c_\phi(X)$.

The next two Theorems 3.3 and 3.4 address the measurement of AP-risk aversion and comparative AP-risk aversion for spectral risk measures. While the technical essentials by and large are known from the dual theory of choice, we for the first time apply these results to spectral risk measures in order to point to inconsistencies in the prevailing literature, where the so-called spectral Arrow-Pratt measure is regularly used as measure of AP-risk aversion.

Theorem 3.3. *SRM-decision makers are AP-risk averse.*

For the proof, see ROELL (1987), Proposition II.2, who for the dual theory of choice proves that a DT-decision maker is AP-risk averse if and only if $v(p) \geq p$ for all $p \in [0, 1]$. In terms of spectral risk measures, this condition corresponds to $\Phi(p) \geq p$ for all $p \in [0, 1]$, which by definition is satisfied for all spectral risk measures, as Φ is a concave cumulative distribution function with $\Phi(0) = 0$ and $\Phi(1) = 1$.

Theorem 3.4. *A SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than a SRM-decision maker with risk spectrum ϕ_2 if and only if $\Phi_1(p) - \Phi_2(p) \geq 0$ for all $p \in [0, 1]$.*

For the proof, see again ROELL (1987), Proposition II.4. For a SRM-decision maker to be more AP-risk averse, the antiderivative of his risk spectrum has to lie above the one of the less AP-risk averse SRM-decision maker on the entire support.

For the subclasses of spectral risk measures introduced in Section 2.2, the respective parameters can be consistently interpreted as being parameters of AP-risk aversion, as the following Theorem 3.5 shows.

Theorem 3.5.

1. *Let α_1 and α_2 be the confidence levels of two CVaR-decision makers with risk spectrum as given in (10). Then the CVaR-decision maker with confidence level α_1 is more AP-risk averse than the CVaR-decision maker with confidence level α_2 if and only if $\alpha_1 \leq \alpha_2$.*

2. *Let a_1 and a_2 be the parameters of two SRM-decision makers with exponential risk spectrum as given in (11). Then the SRM-decision maker with a_1 is more AP-risk averse than the SRM-decision maker with a_2 if and only if $a_1 \geq a_2$.*

3. *Let b_1 and b_2 be the parameters of two SRM-decision makers with power risk spectrum as given in (12). Then the SRM-decision maker with b_1 is more AP-risk averse than the SRM-decision maker with b_2 if and only if $b_1 \leq b_2$.*

The proof is given in the Appendix.

3.3. Comparative AP-risk aversion and the spectral Arrow-Pratt measure

Inspired by the well-known Arrow-Pratt measure from expected utility theory, AP-risk aversion in the framework of spectral risk measures in the literature so far is measured by the so-called spectral Arrow-Pratt measure

$$R_\phi(p) = -\frac{\Phi''(p)}{\phi'(p)} = \frac{\phi'(p)}{\phi(p)}. \tag{14}$$

As examples for this literature approach, see, among many others, the statements by WÄCHTER/MAZZONI (2013), p. 490, "Thus, by defining (14) a local measure of risk aversion in terms of the risk spectrum ϕ (...) is defined." (see also their Examples 1-3), or by BARBI/ROMAGNOLI (2013), p. 8, "(...) the exponential risk measure (ERM), whose weights are based on the exponential utility function, where $k > 0$ is the constant Arrow-Pratt coefficient of absolute risk aversion."

As already shown in Theorem 3.4, it is the difference of the antiderivatives of the risk spectra and not the spectral Arrow-Pratt measure that consistently measures AP-risk aversion. More precisely, the following relationships hold between the two measures:

Theorem 3.6. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers. If $R_{\phi_1}(p) \geq R_{\phi_2}(p)$ for all $p \in [0,1]$, then the SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than the SRM-decision maker with risk spectrum ϕ_2. The converse is not true.*

The proof is given in the Appendix. Theorem 3.6 shows that the terminology of "spectral Arrow-Pratt measure" is misleading: It is not a consistent measure of AP-risk aversion for spectral risk measures as the classical Arrow-Pratt measure is for expected utility theory. The following (counter-)example shows that referring to the spectral Arrow-Pratt measure may classify a SRM-decision maker as being locally more AP-risk averse although he is not. Note that the example is relevant also from a practical perspective, as it is based on exponential and power spectral risk measures, both of which are well-established in the literature on, e.g., quantile-based risk measures in insurance (e.g., DOWD/BLAKE (2006)).

Example 3.7. Assume two SRM-decision makers 1 and 2 with power risk spectrum Φ_1 and exponential risk spectrum Φ_2, respectively,

$$\Phi_1(p) = p^b,\ 0 < b \leq 1, p \in [0,1] \tag{15}$$

$$\Phi_2(p) = \frac{1 - e^{-a \cdot p}}{1 - e^{-a}},\ a > 0, p \in [0,1] \tag{16}$$

(see Figure 4). For $b = 0{,}5$ and $a = 1$, it holds that $\Phi_1(p) \geq \Phi_2(p)$ for all $p \in [0,1]$, so SRM-decision maker 1 is uniformly more AP-risk averse due to Theorem 3.4.

Let us now apply the spectral Arrow-Pratt measure instead, as it is commonly proposed in the literature (e.g., WÄCHTER/MAZZONI (2013) or BARBI/ROMAGNOLI (2013)). We
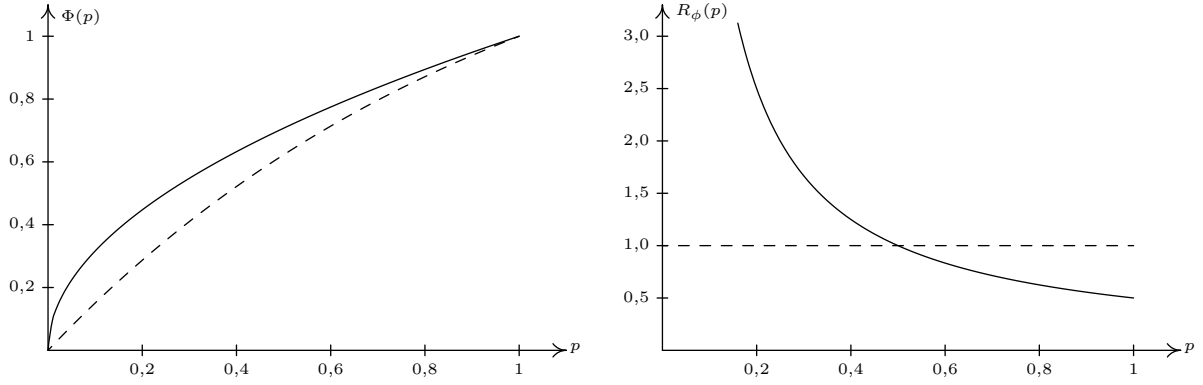
Figure 4: Power (solid) vs. exponential (dashed) risk spectrum and spectral Arrow-Pratt measure

then observe

$$R_{\phi_1}(x) = \frac{1}{2 \cdot p} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1 = R_{\phi_2}(x) \text{ if } \begin{cases} 0 \leq p < 0{,}5 \\ \qquad p = 0{,}5 \\ 0{,}5 < p \leq 1 \end{cases}, \qquad (17)$$

i.e., SRM-decision maker 1 is ranked as being (locally) more AP-risk averse if $p < 0{,}5$, while he is ranked as being (locally) less risk averse if $p > 0{,}5$. This split ranking, however, is misleading as can be seen from the position

$$X = \begin{cases} x_1 & p \\ x_2 & 1 - p \end{cases}, \quad x_1 < x_2, p \in [0, 1], \qquad (18)$$

which covers the cases $p < 0{,}5$ as well as $p > 0{,}5$. In both cases the certainty equivalent of X is given by

$$c_\phi(X) = -\rho_\phi(X) = \Phi(p) \cdot x_1 + (1 - \Phi(p)) \cdot x_2 = \Phi(p) \cdot (x_1 - x_2) + x_2. \qquad (19)$$

Hence, $\Phi_1(p) \geq \Phi_2(p)$ for all $p \in [0, 1]$ implies $c_{\phi_1}(X, p) \leq c_{\phi_2}(X, p)$ for all $p \in [0, 1]$ and confirms that the SRM-decision maker with risk spectrum ϕ_1 is uniformly more AP-risk averse. $\qquad \square$

3.4. Economic relevance of comparative AP-risk aversion: Two standard problems

We now show that the concept of AP-risk aversion is also economically relevant. To this end, we analyze the two economic standard problems of the willingness to pay for insurance and portfolio selection with respect to comparative risk aversion.

Regarding the insurance problem, a measure of risk aversion is consistent if it yields that the more risk averse SRM-decision maker for any risk to insure is willing to pay a greater

premium. Likewise, in the portfolio selection problem between a risk free and a risky asset, a measure of risk aversion is consistent if it yields that the more risk averse SRM-investor always invests less in the risky asset. It will turn out that the two problems are covered by the framework of AP-risk aversion introduced above. Accordingly, for both problems the difference of the antiderivatives of the risk spectra and not the spectral Arrow-Pratt measure will be the consistent measure of risk aversion.

Recall that spectral risk measures either already implicitly represent a risk-reward tradeoff as in the case of exponential and power spectral risk measures, or can be extended towards a (spectral) risk-reward tradeoff as in the case of Conditional Value-at-Risk.

We start with the insurance problem. A SRM-decision maker with deterministic initial wealth w_0 is faced with an additional risk $X \in \mathcal{X}$, $X \neq E(X)$. By signing an insurance contract he can switch from the risky position $w_0 + X$ to the certain position $w_0 - \pi_\phi(w_0, X)$, where $\pi_\phi(w_0, X)$ denotes the insurance premium that the SRM-decision maker is willing to pay to cede X.

By definition, the insurance premium is given by

$$\rho_\phi(w_0 + X) = \rho_\phi(w_0 - \pi_\phi(w_0, X)). \tag{20}$$

Due to the linearity property of spectral risk measures (1), it does not depend on the initial wealth w_0 and shrinks to the negative certainty equivalent of X,

$$\rho_\phi(X) - w_0 = \pi_\phi(w_0, X) - w_0 \Leftrightarrow$$
$$\pi_\phi(w_0, X) = \rho_\phi(X) = -c_\phi(X). \tag{21}$$

Hence, the insurance problem is covered by the framework of AP-risk aversion. Based on (21) and Definition 3.1, we immediately obtain the following comparative static result.

Corollary 3.8. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $\pi_\phi(w_0, X)$ the insurance premium as given in (21). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than SRM-decision maker with risk spectrum ϕ_2.*

2. *$\pi_{\phi_1}(w_0, X) \geq \pi_{\phi_2}(w_0, X)$ for all $X \in \mathcal{X}$, $X \neq E(X)$.*

Due to Corollary 3.8 and Theorem 3.4, it is the non-negative difference of the antiderivatives of the risk spectra and not a greater spectral Arrow-Pratt measure that is necessary and sufficient for consistent comparative static results in the insurance problem. Conversely, when making use of the spectral Arrow-Pratt measure instead, the more AP-risk averse SRM-decision maker may misleadingly be ranked as being locally less risk averse

despite being willing to pay a greater insurance premium for some positions X, as the (counter-)example 3.7 has shown.

Let us proceed with the portfolio selection problem, where w_0 is again the deterministic initial wealth of a SRM-investor. Let further $z \in [0, w_0]$ and $w_0 - z$ be the amounts that are invested in the risk free and a risky asset, respectively. Finally, the returns of the risk free and the risky asset are given by $(r_f - 1)$ and $(r - 1), r \in \mathcal{X}, r \neq E(r), E(r) > r_f$, respectively. The SRM-investor's final wealth reads

$$(w_0 - z) \cdot r + z \cdot r_f. \tag{22}$$

The optimal amount that is invested in the risk free asset is given by

$$z_\phi^*(w_0, r) = \underset{z \in [0, w_0]}{\arg\ \min}\ \rho_\phi((w_0 - z) \cdot r + z \cdot r_f)$$

$$= \underset{z \in [0, w_0]}{\arg\ \min}\ (w_0 - z) \cdot \rho_\phi(r) - z \cdot r_f, \tag{23}$$

where again the linearity property (1) has been used. The corresponding first order condition reads

$$\frac{\partial \rho_\phi(\cdot)}{\partial z} = -\rho_\phi(r) - r_f \tag{24}$$

and yields

$$z_\phi^*(w_0, r) = \begin{cases} w_0 & \rho_\phi(r) > -r_f = \rho_\phi(r_f) \\ 0 & \rho_\phi(r) \leq -r_f = \rho_\phi(r_f) \end{cases}. \tag{25}$$

Spectral risk measures yield an "all or nothing"-decision: Instead of portfolio diversification, either the exclusive investment in the risk free asset, or the exclusive investment in the risky asset is optimal. This result fundamentally differs from classical results obtained for expected utility theory or mean-variance-approaches, and it is a first pitfall of spectral risk measures when they are used for portfolio selection. As has been recently shown by BRANDTNER (2013), non-diversification also prevails in extended settings with more than one risky asset and without the risk free asset. Note further that due to the linearity property (1) the optimal investment does not depend on the initial wealth w_0.

As the amount invested in the risk free assets solely depends on the (negative) certainty equivalent, $\rho_\phi(r) = -c_\phi(r)$, the portfolio selection problem is covered by the framework of AP-risk aversion and due to (25) and Definition 3.1 yields

Corollary 3.9. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $z_\phi^*(w_0, r)$ the optimal amount that is invested in the risk free asset as given in (25). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than SRM-decision maker with risk spectrum ϕ_2.*

2. $z^*_{\phi_1}(w_0, r) \geq z^*_{\phi_2}(w_0, r)$ *for all $r \in \mathcal{X}, r \neq E(r), E(r) > r_f$.*

Due to Corollary 3.9 and Theorem 3.9 it is again the non-negative difference of the antiderivatives of the risk spectra and not a greater spectral Arrow-Pratt measure that is necessary and sufficient for consistent comparative static results in the portfolio selection problem. Employing the spectral Arrow-Pratt measure instead may be misleading as has been shown by the (counter-)example 3.7.

4. Comparative R-risk aversion: The case of random initial wealth

4.1. Definitions

The framework of ARROW (1965) and PRATT (1964) covers decision situations where the initial wealth is deterministic and where the risk of the final wealth can be completely eliminated. ROSS (1981) has extended this framework by assuming that the initial wealth is random and that the risk of the final wealth can only be eliminated partly. Accordingly, the framework of ROSS (1981) addresses the shift from "more" to "less" risk, which appears to be the more realistic setting, in particular from the point of view of the recent financial crises.

In order to model increasing risk, ROSS (1981) has made use of ROTHSCHILD/STIGLITZ (1970)'s concept of a mean preserving spread.

Definition 4.1. *A random variable Z is called more risky than a random variable X if $Z = X + Y$ with $E(Y|X) = 0$.*

Theorem 4.2. *The following holds:*

1. *$Z = X + Y$ with $E(Y|X) = 0 \Rightarrow Z$ is a mean preserving spread of X, i.e., $\int\limits_{-\infty}^{t} F_Z(x) - F_X(x)\mathrm{d}x \geq 0$ for all $t \in \mathbb{R}$ and $\int\limits_{-\infty}^{\infty} F_Z(x) - F_X(x)\mathrm{d}x = 0$.*

2. *Z is a mean preserving spread of $X \Rightarrow \exists Y$ such that $Z \stackrel{d}{=} X + Y$ and $E(Y|X) = 0$.*

Note that the proposed form of increasing risk is consistent with spectral risk measures, as all SRM-decision makers reject any mean preserving spread (for a proof, see ADAM ET AL. (2008), Appendix A or LEITNER (2005)).

In order to capture a reduction of risk by switching from Z to X, ROSS (1981) has introduced the incremental risk premium.

Definition 4.3. *For the positions $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$, the incremental risk premium, $RP(X, Y)$, is defined by $RP(X, Y) = c(X) - c(X + Y)$.*

The incremental risk premium indicates the payment a decision maker is willing to make to avoid Y and to switch from Z to X. Based on the incremental risk premium, R-risk aversion and comparative R-risk aversion are defined as follows.

Definition 4.4.

a) *A decision maker is said to be R-risk averse if $RP(X, Y) \geq 0$ for all $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$.*

b) *A decision maker 1 is said to be more R-risk averse than a decision maker 2 if $RP_1(X, Y) \geq RP_2(X)(\geq 0)$ for all $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$.*

A R-risk averse decision maker is willing to make a non-negative payment to avoid increasing risk. For the more R-risk averse decision maker, this payment is greater than for the less R-risk averse decision maker.

4.2. Comparative R-risk aversion for spectral risk measures

We now address the measurement of R-risk aversion for spectral risk measures. To this end, we first specify incremental risk premium.

Theorem 4.5. *For the positions $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$, the incremental risk premium of a SRM-decision maker with risk spectrum ϕ, $RP_\phi(X, Y)$, is given by*

$$RP_\phi(X, Y) = \rho_\phi(X + Y) - \rho_\phi(X). \tag{26}$$

The proof directly follows from the identity $c_\phi(X) = -\rho_\phi(X)$ (see Theorem 3.2).

The following Theorems 4.6 and 4.7 address the measurement of R-risk aversion and comparative R-risk aversion for spectral risk measures. Note that the technical essentials again are based on previous results from the dual theory of choice; their implications for the issue of decision making under spectral risk measures, however, have not been made explicit yet.

Theorem 4.6. *SRM-decision makers are R-risk averse.*

For the proof, see YAARI (1987), Theorem 2, who for the dual theory of choice proves that a DT-decision maker is R-risk averse if and only if $v(p)$ is concave on the entire support. In terms of spectral risk measures, this condition corresponds to $\Phi(p)$ being concave on the entire support, which is satisfied by definition. Note that although the technical requirements for AP-risk aversion (i.e., $\Phi(p) \geq p$, see Theorem 3.4) and R-risk aversion (i.e., $\Phi(p)$ concave, see Theorem 3.9) differ, they are both satisfied for spectral risk measures by definition. Consequently, for spectral risk measures, AP-risk aversion and R-risk aversion are equivalent concepts.

Theorem 4.7. *A SRM-decision maker with risk spectrum ϕ_1 is more R-risk averse than a SRM-decision maker with risk spectrum ϕ_2 if and only if $\Phi_1(p) - \Phi_2(p)$ is non-negative and concave on the entire support.*

For the proof, see ROELL (1987), Proposition II.5, again for the dual theory of choice. For a SRM-decision maker to be more R-risk averse, two conditions have to be satisfied. The first condition requires that the antiderivative of the risk spectrum of the more R-risk averse SRM-decision maker lies above the one of the SRM-decision maker who is less R-risk averse, $\Phi_1(p) - \Phi_2(p) \geq 0$. The second, and additional, condition requires that the difference of the antiderivatives of the risk spectra is concave on the entire support. Hence, greater R-risk aversion implies greater AP-risk aversion, but the converse it not true.

4.3. Comparative R-risk aversion: Three subclasses of spectral risk measures

We now specify the insights from Section 4.2 with respect to Conditional Value-at-Risk, and exponential and power spectral risk measures. While these relevant subclasses could be completely ordered with respect to AP-risk aversion, we show by means of a simple counter-example that under R-risk aversion this useful property fails.

4.3.1. A (counter-)example

Building on Theorem 4.7, we argue by contradiction and assume that there exists some $\bar{p} < 1$ such that $\Phi_1(p) - \Phi_2(p)$ is decreasing and convex on $p \in [\bar{p}, 1]$ (see Figure 5).[1] Then one can construct a binary random variable for which the more AP-risk averse SRM-decision maker is less R-risk averse.

Let X and $X + Y$ be two risky positions with

$$X = \begin{cases} 0 & q_1 \\ \frac{1-q_2}{1-q_1} & 1 - q_1 \end{cases}, \; X + Y = \begin{cases} 0 & q_2 \\ 1 & 1 - q_2 \end{cases}, \; 0 \leq q_1 < q_2 \leq 1; \tag{27}$$

---

[1] A similar but more general counter-example can be constructed by setting the less restrictive assumption that $\Phi_1(p) - \Phi_2(p)$ is convex on some interval $p \in [\underline{p}, \overline{p}] \subseteq [0, 1]$. For any of our subclasses of spectral risk measures, however, it will turn out that $\Phi_1(p) - \Phi_2(p)$ is decreasing and convex on some interval $[\bar{p}, 1]$, and for the sake of consistency we set up the corresponding assumptions in our counter-example.
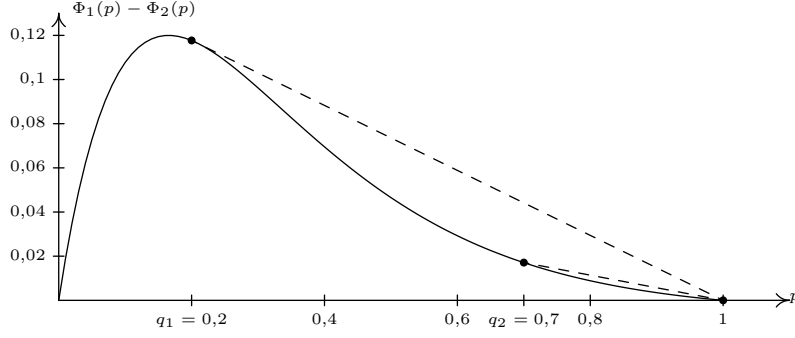
Figure 5: Comparative R-risk aversion and difference of the antiderivatives of the risk spectra

$X + Y$ is constructed from X by a mean preserving spread. If SRM-decision maker 1 is more R-risk averse than SRM-decision maker 2, the incremental risk premiums (see Definition 4.4) fulfill

$$RP_{\phi_1}(X,Y) - RP_{\phi_2}(X,Y) \geq 0 \Leftrightarrow \frac{\Phi_1(q_2) - \Phi_2(q_2)}{1 - q_2} \geq \frac{\Phi_1(q_1) - \Phi_2(q_1)}{1 - q_1}. \tag{28}$$

However, as $\Phi_1(p) - \Phi_2(p)(\geq 0)$ has been assumed to be decreasing and convex on $[\bar{p}, 1]$, choosing $q_1, q_2 \in [\bar{p}, 1]$ yields

$$\frac{\Phi_1(q_2) - \Phi_2(q_2)}{1 - q_2} < \frac{\Phi_1(q_1) - \Phi_2(q_1)}{1 - q_1} \Leftrightarrow RP_{\phi_1}(X,Y) - RP_{\phi_2}(X,Y) < 0, \tag{29}$$

a contradiction (see Figure 5).

4.3.2. Examples

We now analyze the relationship between the popular subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures and R-risk aversion.

To start with Conditional Value-at-Risk, the difference of the antiderivatives of the risk spectra is given by

$$\Phi_{\alpha_1}(p) - \Phi_{\alpha_2}(p) = \begin{cases} \frac{p}{\alpha_1} - \frac{p}{\alpha_2} & 0 \leq p \leq \alpha_1 \\ 1 - \frac{p}{\alpha_2} & \alpha_1 < p \leq \alpha_2 \\ 0 & \alpha_2 < p \leq 1 \end{cases} \tag{30}$$

and is convex on $[\alpha_1, 1]$ for any two confidence levels $0 \leq \alpha_1 < \alpha_2 \leq 1$. Figure 6 illustrates the case for $\alpha_1 = 0{,}4$ and $\alpha_2 = 0{,}8$.

Accordingly, for any two confidence levels $\alpha_1 < \alpha_2$ one can construct the following (counter-)example where the less AP-risk averse CVaR-decision maker, at the same time, is more R-risk averse and exhibits a greater incremental risk premium.
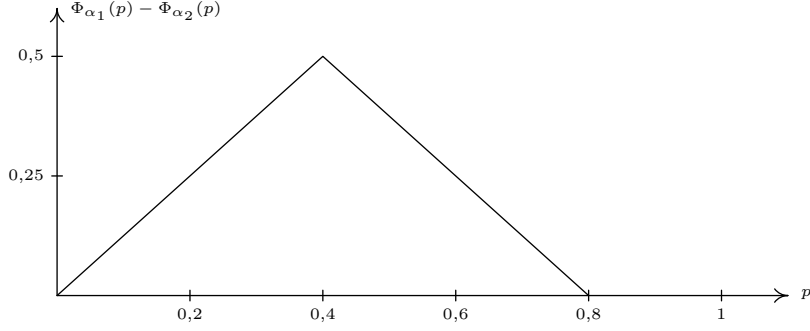
16

Figure 6: Difference of the antiderivatives of the risk spectra for Conditional Value-at-Risk and $\alpha_1 = 0.4$ and $\alpha_2 = 0.8$

Example 4.8. Let α_1 and α_2, $\alpha_1 < \alpha_2$, be the confidence levels of two CVaR-decision makers. Let further the two risky positions from the general (counter-)example (27) with $q_1 = \alpha_1$ and $q_2 = \alpha_2$ be given by

$$X = \begin{cases} 0 & \alpha_1 \\ \frac{1-\alpha_2}{1-\alpha_1} & 1 - \alpha_1 \end{cases}, X + Y = \begin{cases} 0 & \alpha_2 \\ 1 & 1 - \alpha_2 \end{cases}. \tag{31}$$

We then have

$$RP_{\alpha_1}(X,Y) = CVaR_{\alpha_1}(X+Y) - CVaR_{\alpha_1}(X) = 0 - 0 = 0 \tag{32}$$

$$RP_{\alpha_2}(X,Y) = CVaR_{\alpha_2}(X+Y) - CVaR_{\alpha_2}(X) = 0 + \frac{1-\alpha_2}{1-\alpha_1} \cdot \frac{\alpha_2 - \alpha_1}{\alpha_2} > 0. \tag{33}$$

Contrary to intuition, the more AP-risk averse CVaR-decision maker (with confidence level α_1) is not willing to make a positive payment to reduce risk by switching from $X + Y$ to X, as he only takes into account the $\alpha_1 \cdot 100\%$ worst outcomes, which for both positions correspond to zero. The less AP-risk averse CVaR-decision maker (his confidence level is at $\alpha_2 > \alpha_1$) for the position X also (partly) takes into account the states of the world where the outcome is positive, so he is willing to make a positive payment to reduce his risk. $\square$

The next popular case is the subclass of exponential spectral risk measures. Concerning comparative R-risk aversion, the following holds.

Theorem 4.9. *Let $\{\rho_{\phi_a}, a > 0\}$ be the subclass of exponential spectral risk measures as given in (10). Let further $a_2 < a_1$, i.e., SRM-decision maker 2 is less AP-risk averse than SRM-decision maker 1, and let $\bar{a} = 1.5937$.*

1. *For any two parameters $a_2 < a_1 < \bar{a}$, $\Phi_{a_1}(p) - \Phi_{a_2}(p)$ is non-negative and concave on $[0,1]$.*

2. *For any two parameters $a_1 > a_2 > \bar{a}$, $\Phi_{a_1}(p) - \Phi_{a_2}(p)$ is non-negative and convex on $\left[\bar{p} = \frac{1}{a_1 - a_2} \cdot \ln\left(\frac{a_1^2 \cdot (1 - e^{-a_2})}{a_2^2 \cdot (1 - e^{-a_1})} \right), 1 \right]$.*

17

3. For any two parameters $a_2 < \bar{a} < a_1$ with $\frac{e^{a_1}-1}{a_1^2} < \frac{e^{a_2}-1}{a_2^2}$, $\Phi_{a_1}(p) - \Phi_{a_2}(p)$ is non-negative and convex on $\left[\bar{p} = \frac{1}{a_2 - a_1} \cdot \ln\left(\frac{a_2^2 \cdot (1 - e^{-a_1})}{a_1^2 \cdot (1 - e^{-a_2})} \right), 1 \right]$.

The proof is given in the Appendix. In the first case, both SRM-decision makers are assumed to be slightly AP-risk averse, i.e., $a < 1{,}5937$. Then a greater a consistently represents greater AP- and R-risk aversion. In the second case, both SRM-decision makers are assumed to be strongly AP-risk averse, i.e., $a > 1{,}5937$. Then the decision maker with $a_1 > a_2$ is uniformly more AP-risk averse, but he is not uniformly more R-risk averse. By contrast, by making use of (27), for any two a_1 and a_2 one can construct a (counter-)example where the more AP-risk averse SRM-decision maker exhibits a smaller incremental risk premium than the less AP-risk averse SRM-decision maker. We illustrate this case in more detail in Example 4.10 below. Finally, when considering a slightly and a strongly AP-risk averse SRM-decision maker, the results are ambiguous. If the difference $a_1 - a_2 > 0$ is sufficiently large, the more AP-risk averse decision maker is not uniformly more R-risk averse, while he is if the difference is sufficiently small.

Example 4.10. Let ϕ_a be the exponential risk spectrum of a SRM-decision maker as given in (10). Let further be X and $X + Y$ two risky positions as in (27) with $q_1 = 0{,}5$ and $q_2 = 0{,}75$

$$X = \begin{cases} 0 & 0{,}5 \\ 0{,}5 & 0{,}5 \end{cases}, \quad X + Y = \begin{cases} 0 & 0{,}75 \\ 1 & 0{,}25 \end{cases}. \tag{34}$$

Now assume that the degree of AP-risk aversion is increased by increasing a. It holds that (see (29))

$$\frac{\partial \Phi_a(q_1)}{\partial a} \cdot \frac{1}{1 - q_1} \left\{ \begin{array}{c} \leq \\ > \end{array} \right\} \frac{\partial \Phi_a(q_2)}{\partial a} \cdot \frac{1}{1 - q_2} \text{ if } \left\{ \begin{array}{c} a \leq 2{,}44 \\ a > 2{,}44 \end{array} \right. . \tag{35}$$

Accordingly, for $a \leq 2{,}44$, the incremental risk premium is increasing in a. However, for $a > 2{,}44$ the incremental risk premium is decreasing in a: Despite the SRM-decision maker becomes more AP-risk averse, his incremental risk premium is decreasing. Figure 7 illustrates these results. □

We finally analyze the subclass of power spectral risk measures. Here the following holds.

Theorem 4.11. Let $\{\rho_{\phi_b}, 0 < b \leq 1\}$ be the subclass of power spectral risk measures as given in (11). Let further $b_1 < b_2$, i.e., SRM-decision maker 2 is less AP-risk averse than SRM-decision maker 1, and let $\bar{b} = 0{,}5$.

1. For any two parameters $\bar{b} < b_1 < b_2$, $\Phi_{b_1}(p) - \Phi_{b_2}(p)$ is non-negative and concave on $[0, 1]$.
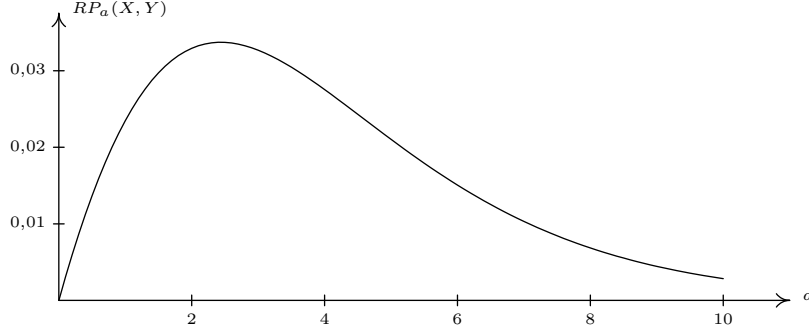
18

Figure 7: Incremental risk premium for exponential spectral risk measures

2. *For any two parameters $b_1 < b_2 < \bar{b}$, $\Phi_{b_1}(p) - \Phi_{b_2}(p)$ is non-negative and convex on*
   $\left[\bar{p} = \left(\frac{b_2 \cdot (1-b_2)}{b_1 \cdot (1-b_1)}\right)^{\frac{1}{b_1 - b_2}}, 1\right]$.

3. *For $b_1 < \bar{b} < b_2$, $\Phi_{b_1}(p) - \Phi_{b_2}(p)$ is non-negative and convex on $\left[\bar{p} = \left(\frac{b_2 \cdot (1-b_2)}{b_1 \cdot (1-b_1)}\right)^{\frac{1}{b_1 - b_2}}, 1\right]$*
   if $b_2 < 1 - b_1$.

The proof is given in the Appendix. Again, if both SRM-decision makers are slightly AP-risk averse, i.e., $b > 0{,}5$, a smaller b consistently represents greater AP- and R-risk aversion. By contrast, if both SRM-decision makers are strongly AP-risk averse, i.e., $b < 0{,}5$, the SRM-decision maker with $b_1 < b_2$ is uniformly more AP-risk averse, while he is not uniformly more R-risk averse. The counter-example (27) again allows to illustrate this general incompatibility between AP-risk aversion and R-risk aversion for SRM-decision makers. Finally, when considering a slightly and a strongly AP-risk averse SRM-decision maker, the results are ambiguous. The following example is given for illustration.

Example 4.12. Let ϕ_b be the power risk spectrum of a SRM-decision maker as given in (11). As in the previous Example 4.10, let X and $X + Y$ be two risky positions as in (27) with $q_1 = 0{,}5$ and $q_2 = 0{,}75$

$$X = \begin{cases} 0 & 0{,}5 \\ 0{,}5 & 0{,}5 \end{cases}, \quad X + Y = \begin{cases} 0 & 0{,}75 \\ 1 & 0{,}25 \end{cases}. \tag{36}$$

Now assume that the degree of AP-risk aversion is increased by decreasing b. It holds that (see (29))

$$\frac{\partial \Phi_b(q_1)}{\partial b} \cdot \frac{1}{1 - q_1} \begin{Bmatrix} \leq \\ > \end{Bmatrix} \frac{\partial \Phi_b(q_2)}{\partial b} \cdot \frac{1}{1 - q_2} \text{ if } \begin{cases} b \geq 0{,}4593 \\ b < 0{,}4593 \end{cases}. \tag{37}$$

Accordingly, for $b \geq 0{,}4593$, the incremental risk premium is decreasing in b, i.e., decreasing with decreasing AP-risk aversion. However, for $b < 0{,}4593$ the incremental risk premium is increasing in b, i.e., increasing with decreasing AP-risk aversion: Despite the SRM-decision
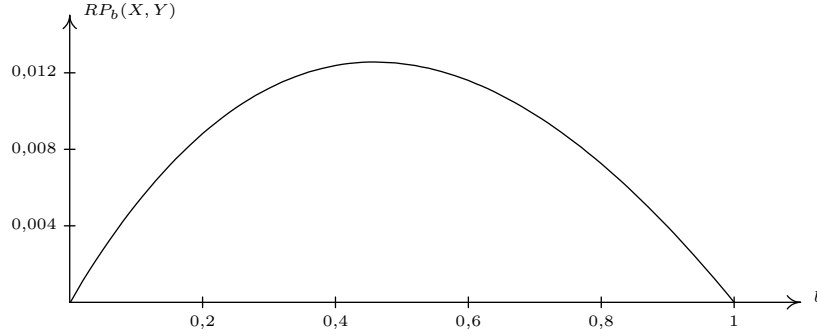
Figure 8: Incremental risk premium for power spectral risk measures

maker becomes less AP-risk averse, his incremental risk premium is increasing. Figure 8 illustrates these results. $\qquad\square$

4.3.3. R-risk aversion vs. completeness: A non-consistency result

In this section, we show that a subclass of spectral risk measures that covers any degree of risk aversion between risk neutrality and infinite risk aversion cannot be completely ordered with respect to R-risk aversion.

The property of completeness is defined as follows.

Definition 4.13. *Let* $\{\rho_{\phi_\theta},\ \theta \in (\underline{\theta}, \overline{\theta})\}$, *be a one-parameter family of spectral risk measures. Let further* X *be a uniformly distributed random variable on* $[0, 1]$. $\{\rho_{\phi_\theta}\}$ *is said to be complete if* $\rho_{\phi_\theta}(X)$ *can take any value in* $[-0{,}5, 0]$.

A one-parameter family of spectral risk measures is complete if it covers any degree of risk aversion between risk neutrality and infinite risk aversion. If $\rho_{\phi_\theta}(X) = -0{,}5 = -E(X)$, the SRM-decision maker is risk neutral. If $\rho_{\phi_\theta}(X) = 0$, the SRM-decision maker is infinitely risk averse, as in this case the spectral risk measure corresponds to the negative maximum loss, $\rho_{\phi_\theta}(X) = -\inf\{X(\omega) : \omega \in \Omega\}$. Moreover, whatsoever spectral risk $\bar{\rho} \in [-0{,}5, 0]$ the SRM-decision maker wants to assign to X, there exists a corresponding parameter $\theta = \theta(\bar{\rho})$ that yields this risk. Note that the subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures are complete.

Although completeness is a desirable property of a family of spectral risk measures, it, at the same time, contradicts with R-risk aversion:

Theorem 4.14. *Let* $\{\rho_{\phi_\theta},\ \theta \in (\underline{\theta}, \overline{\theta})\}$ *be a one-parameter family of spectral risk measures that satisfies the completeness property as given in Definition 4.13. Then there exists at least one pair of risk spectra* $\theta_1, \theta_2 \in (\underline{\theta}, \overline{\theta})$ *and* $\underline{p}, \overline{p} \in [0, 1]$ *such that* $\Phi_{\theta_1}(p) - \Phi_{\theta_2}(p)$ *is convex on* $[\underline{p}, \overline{p}] \subseteq [0, 1]$.

For the proof it is sufficient to note that the antiderivative of the risk spectrum of the negative maximum loss is given by $\Phi_{ML}(p) = 1, p \in [0, 1]$, and that $\Phi_{ML}(p) - \Phi_{\theta_2}(p)$ is convex on $[0, 1]$ for any $\theta_2 \in (\underline{\theta}, \overline{\theta})$.

Theorem 4.14 further confirms our observations from Section 4.3.2 that strongly AP-risk averse SRM-decision makers face inconsistencies with R-risk aversion: In the extreme case of infinite AP-risk aversion, the (negative) Maximum Loss decision maker has the risk spectrum $\Phi_{ML}(p) = 1, p \in [0, 1]$, and the difference $\Phi_{ML}(p) - \Phi_{\theta_2}(p)$ is convex on $[0, 1]$ for any $\theta_2 \in (\underline{\theta}, \overline{\theta})$. Conversely, in the extreme case of zero AP-risk aversion, the (negative) Mean-decision maker has the risk spectrum $\Phi_\mu(p) = p, p \in [0, 1]$, so the difference $\Phi_{\theta_2}(p) - \Phi_\mu(p)$ is concave on $[0, 1]$ for any $\theta_2 \in (\underline{\theta}, \overline{\theta})$.

4.4. Economic relevance of comparative R-risk aversion: Two standard problems

In order to demonstrate the economic relevance of the concept of R-risk aversion, we again analyze an insurance problem and a portfolio selection problem. Unlike the our previous study of these problems in Section 3.4, the decision maker's initial wealth now is assumed to be random, and the risk of the final wealth can only be eliminated partly.

In the insurance problem with some random initial wealth, a measure of risk aversion is consistent if it yields that the more risk averse SRM-decision maker is willing to pay a greater insurance premium for any reduction in risk in the sense of Definition 4.1. Likewise, in a portfolio selection problem with a risk free and a risky asset in the presence of a random initial wealth, a measure of risk aversion is consistent if it yields that the more risk averse SRM-investor always invests less in the risky asset. It will turn out that the two problems are covered by the framework of R-risk aversion introduced above. Accordingly, spectral risk measures will regularly yield counterintuitive results when being applied to the insurance problem and the portfolio selection problem, respectively.

Once again recall that spectral risk measures either already implicitly represent a risk-reward tradeoff as in the case of exponential and power spectral risk measures, or can be extended towards a (spectral) risk-reward tradeoff as in the case of Conditional Value-at-Risk.

We start with the insurance problem. A SRM-decision maker with random initial wealth $w_0 + X \in \mathcal{X}, X \neq E(X)$, is faced with an additional risk $Y \in \mathcal{X}, Y \neq E(Y), E(Y|X) = 0)$. By signing an insurance contract, he can switch from the risky position $w_0 + X + Y$ to the risky position $w_0 + X - \pi_\phi(X, Y)$, where $\pi_\phi(X, Y)$ denotes the insurance premium that the SRM-decision maker is willing to pay to cede Y.

By definition, the insurance premium satisfies

$$\rho_\phi(w_0 + X + Y) = \rho_\phi(w_0 + X - \pi_\phi(X, Y)). \tag{38}$$

21

Due to the linearity property of spectral risk measures (1), it does not depend on the initial wealth w_0 and is given by the incremental risk premium

$$\rho_\phi(X+Y) - w_0 = \rho_\phi(X) - w_0 + \pi_\phi(X,Y) \Leftrightarrow$$
$$\pi_\phi(X,Y) = \rho_\phi(X+Y) - \rho_\phi(X) = RP_\phi(X,Y). \tag{39}$$

Hence, the insurance problem is covered by the framework of R-risk aversion. Based on (39) and Definition 4.4, we immediately obtain the following comparative static result.

Theorem 4.15. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $\pi_\phi(X,Y)$ the insurance premium as given in (39). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more R-risk averse than SRM-decision maker with risk spectrum ϕ_2.*

2. *$\pi_{\phi_1}(X,Y) \geq \pi_{\phi_2}(X,Y)$ for all $w_0 + X, Y, X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$.*

According to Theorems 4.7 and 4.15, a non-negative and concave difference of the antiderivatives of the risk spectra is necessary and sufficient for consistent comparative static results in the insurance problem.[2] Accordingly, Conditional Value-at-Risk, and exponential and power spectral risk measures regularly yield counterintuitive results as soon as they are applied to questions of optimal insurance. In all three subclasses, the willingness to pay for insurance is non-monotonic in the respective AP-risk aversion parameters (see again the (counter-)examples 4.8, 4.10, and 4.12). Especially, for strongly AP-risk averse SRM-decision makers one can always find (counter-)examples where the willingness to pay for insurance is decreasing though they exhibit increasing AP-risk aversion.

We proceed with the portfolio selection problem. Let w_0 denote the SRM-decision maker's deterministic initial wealth and $X \in \mathcal{X}, X \neq E(X)$ be his random initial wealth. The amounts $z \in [0, w_0]$ and $w_0 - z$, are invested in the risk free and a risky asset, respectively. Finally, the returns of the risk free and the risky asset are given by $(r_f - 1)$ and $(r - 1), r \in \mathcal{X}, r \neq E(r), E(r) > r_f, r_0 := r - E(r), E(r_0|X) = 0$, respectively. The SRM-investor's final wealth is given by

$$X + (w_0 - z) \cdot r + z \cdot r_f. \tag{40}$$

---

[2]Note that due to the property of translation invariance, Theorem 4.15 also holds for $E(Y) \neq 0$, when assuming $E(Y - E(Y)|X) = 0$.

The optimal amount that is invested in the risk free asset is given by

$$z_\phi^*(X, r) = \underset{z \in [0, w_0]}{\arg\min} \; \rho_\phi(X + (w_0 - z) \cdot r + z \cdot r_f)$$

$$= \underset{z \in [0, w_0]}{\arg\min} \; \rho_\phi(X + (w_0 - z) \cdot r_0) - (w_0 - z) \cdot E(r) - z \cdot r_f. \qquad (41)$$

The linearity in (41) again is induced by the linearity property (1) of spectral risk measures. The corresponding first order condition reads

$$\frac{\partial \rho_\phi(\cdot)}{\partial z} = \frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) + (E(r) - r_f) \overset{!}{=} 0 \Leftrightarrow$$

$$\frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) \overset{!}{=} \rho_\phi(E(r) - r_f) \qquad (42)$$

The left-hand side of (42) is covered by the framework of R-risk aversion and thus yields the following Theorem 4.16 (the proof is given in the Appendix).

Theorem 4.16. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $z_\phi^*(X, r)$ the optimal risk free investment as given by (42). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more R-risk averse than SRM-decision maker with risk spectrum ϕ_2.*

2. *$z_{\phi_1}^*(X, r) \geq z_{\phi_2}^*(X, r)$ for all $X, r \in \mathcal{X}, r \neq E(r), X \neq E(X), E(r) > r_f, r_0 = r - E(r), E(r_0|X) = 0$.*

According to Theorems 4.7 and 4.16, a non-negative and concave difference of the antiderivatives of the risk spectra is necessary and sufficient for consistent comparative static results in the portfolio selection problem. Unfortunately, for Conditional Value-at-Risk, and exponential and power spectral risk measures this difference is regularly convex. We thus obtain counterintuitive results in the portfolio selection problem, as the following (counter-)examples show.

Example 4.17. Let α be the confidence level of a CVaR-decision maker. Let further be $w_0, X, r_0, E(r)$ and r_f be given by

$$w_0 = 80, \; X = \begin{cases} 8 & 0{,}1 \\ 8 & 0{,}1 \\ 10 & 0{,}2 \\ 10 & 0{,}6 \end{cases}, \; r_0 = \begin{cases} 0 & 0{,}1 \\ 0 & 0{,}1 \\ -0{,}05 & 0{,}2 \\ 0{,}0167 & 0{,}6 \end{cases}, \; E(r) = 1{,}06, \; r_f = 1{,}05. \qquad (43)$$

The optimal investment in the risk free asset, z_α^*, as a function of the confidence level α is given in Figure 9.
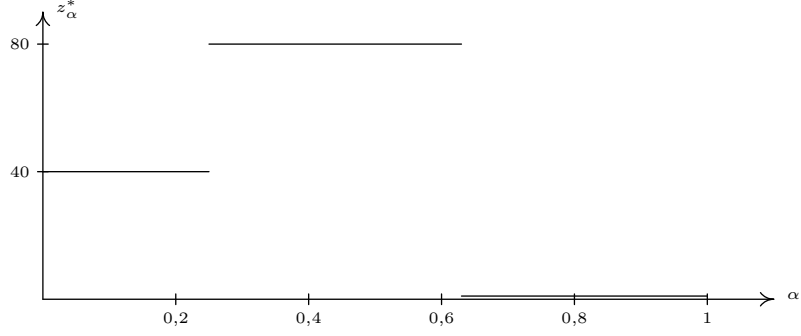
Figure 9: The optimal investment in the risk free asset as a function of the confidence level α

The optimal investment exhibits two counter-intuitive characteristics. First, we again observe a tendency towards corner solutions. Despite the entire investment opportunity set $z \in [0, 80]$ is efficient in that none of the investment opportunities is dominated in terms of a mean preserving spread, the set of optimal solutions only consists of $z_\alpha^* = \{0, 40, 80\}$. The reason is once again the linearity property (1), which has already been responsible for the all-or-nothing decisions in the portfolio selection problem with deterministic initial wealth.

Second, and as one would expect, when decreasing the CVaR-investor's confidence level α, the optimal investment in the risk free asset is initially increasing: We have $z_\alpha^* = 0$ for $\alpha \in (0{,}63, 1]$ and $z_\alpha^* = 80$ for $\alpha \in (0{,}25, 0{,}63)$, i.e., the optimum jumps from the exclusive investment in the risky asset to the exclusive investment in the risk free asset as the confidence level falls below $\alpha = 0{,}63$. However, when the confidence level falls below $\alpha = 0{,}25$, the investment in the risk free asset is decreasing again. Instead of investing in the risk free asset exclusively, the CVaR-investor for $\alpha < 0{,}25$ now invests $z^* = 40$ in the risk free asset and $w_0 - z^* = 80 - 40 = 40$ in the risky asset. This is clearly counter-intuitive and follows from the piecewise convex difference of the antiderivatives of the risk spectra of Conditional Value-at-Risk. $\qquad\square$

Example 4.18. Let a be the parameter of a SRM-decision maker with exponential risk spectrum. Let further be $w_0, X, r_0, E(r)$ and r_f be given by

$$
w_0 = 80, \quad X = \begin{cases} 8 & 0{,}1 \\ 8 & 0{,}1 \\ 10 & 0{,}6 \\ 10 & 0{,}2 \end{cases}, \quad r_0 = \begin{cases} 0 & 0{,}1 \\ 0 & 0{,}1 \\ -0{,}05 & 0{,}6 \\ 0{,}15 & 0{,}2 \end{cases}, \quad E(r) = 1{,}06, \quad r_f = 1{,}043. \quad (44)
$$

The optimal investment in the risk free asset, z_a^*, as a function of the parameter a is given in Figure 4.18. Again, we observe a tendency towards corner solutions, as the set of optimal solutions only consists of $z_a^* = \{0, 40, 80\}$. Moreover, the investment in the risk free asset is
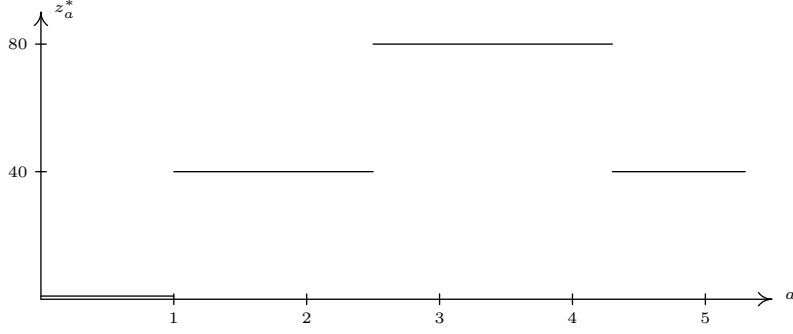
24

Figure 10: The optimal investment in the risk free asset as a function of the parameter a
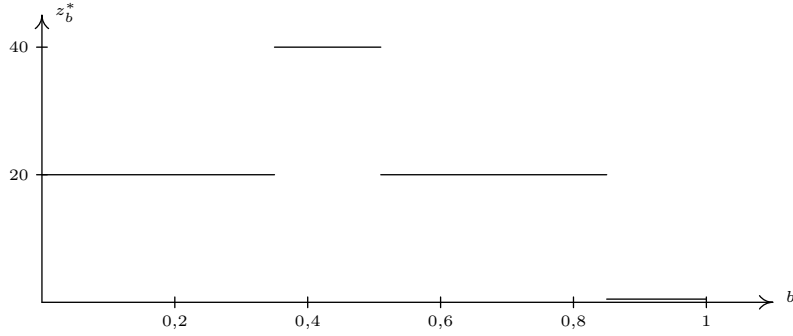

Figure 11: The optimal investment in the risk free asset as a function of the parameter b

decreasing as the SRM-decision maker becomes strongly AP-risk averse: For $a \in (2{,}5, 4{,}3)$, the SRM-decision maker invests his entire wealth in the risk free asset, i.e. $z_a^* = 80$, while for $a \in [4{,}3, \infty)$ he switches back to a 50/50-investment, i.e. $z_a^* = w_0 - z_a^* = 40$. $\qquad \square$

Example 4.19. Let b be the parameter of a SRM-decision maker with power risk spectrum. Let further be $w_0, X, r_0, E(r)$ and r_f be given by

$$
w_0 = 40, \quad X = \begin{cases} 8 & 0{,}1 \\ 8 & 0{,}1 \\ 10 & 0{,}6 \\ 10 & 0{,}2 \end{cases}, \quad r_0 = \begin{cases} 0 & 0{,}1 \\ 0 & 0{,}1 \\ -0{,}10 & 0{,}6 \\ 0{,}30 & 0{,}2 \end{cases}, \quad E(r) = 1{,}06, \quad r_f = 1{,}047. \quad (45)
$$

The optimal investment in the risk free asset, z_b^*, as a function of the parameter b is given in Figure 11. Again, the set of optimal solutions only consists of three elements, namely $z_b^* = \{0, 20, 40\}$. Moreover, the investment in the risk free asset is decreasing as the SRM-decision maker becomes strongly AP-risk averse: For $b \in (0{,}35, 0{,}51)$, the SRM-decision maker invests his entire wealth in the risk free asset, i.e. $z_a^* = 40$, while for $b \in (0, 0{,}35]$ he switches back to a 50/50-investment, i.e. $z_a^* = w_0 - z_a^* = 20$. $\qquad \square$

5. Conclusions

In this paper, we have studied the concepts of comparative risk aversion following Arrow (1965) and Pratt (1964) on the one hand, and Ross (1981) on the other hand, together with their implications for the willingness to pay for insurance and portfolio selection in the context of spectral risk measures.

In the framework following Arrow (1965) and Pratt (1964), we have shown that the difference of the antiderivatives of the risk spectra and not the spectral Arrow-Pratt measure is the consistent measure of AP-risk aversion. Conversely, when applying the spectral Arrow-Pratt measure instead as is done in the literature regularly, the less AP-risk averse SRM-decision maker may be ranked as being more AP-risk averse although he ist not. We have further shown that the framework of Arrow (1965) and Pratt (1964) covers the standard economic problems of the willingness to pay for insurance and portfolio selection. Consequently, the spectral Arrow-Pratt measure cannot be applied to these problems either. If one still does, a SRM-decision maker with a greater spectral Arrow-Pratt measure may only be willing to pay less for insurance or to invest more in the risky asset than a SRM-decision maker with a smaller spectral Arrow-Pratt measure.

In the framework following Ross (1981), we have shown that the difference of the antiderivatives of the risk spectra has to be non-negative and concave on the entire support in order to provide consistent comparative static results with respect to R-risk aversion. Neither Conditional Value-at-Risk, nor exponential and power spectral risk measures satisfy these requirements. Accordingly, these three subclasses cannot be completely ordered with respect to R-risk aversion. We further have provided a general non-consistency result for spectral risk measures and R-risk aversion.

As a consequence, these subclasses of spectral risk measures also exhibit counter-intuitive comparative static results with respect to the respective AP-risk aversion parameters, both in the insurance problem and the portfolio selection problem: In the insurance problem, the willingness to pay for insurance may be decreasing with increasing AP-risk aversion. Likewise, in the portfolio selection problem, the investment in the risky asset may be increasing with increasing AP-risk aversion. This is especially the case when SRM-decision makers are assumed to be strongly AP-risk averse.

The paper gives directions for future research. One open comparative statics question, for example, is how increases in the risk of the underlying random variable affect the willingness pay for insurance or the optimal investment in the risk free asset for different spectral risk measures or at different levels of risk aversion. Eeckhoudt et al. (1991) for expected utility theory have shown that demand for insurance may be decreasing when increasing the risk of the underlying random variable. It might be interesting to see whether similar results also prevail for spectral risk measures.

A. Proof of Theorem 3.5

For Conditional Value-at-Risk, the difference of the antiderivatives of the risk spectra

$$\Phi_{\alpha_1}(p) - \Phi_{\alpha_2}(p) = \begin{cases} \frac{p}{\alpha_1} - \frac{p}{\alpha_2} & 0 \leq p \leq \alpha_1 \\ 1 - \frac{p}{\alpha_2} & \alpha_1 < p \leq \alpha_2 \\ 0 & \alpha_2 < p \leq 1 \end{cases} \tag{46}$$

for any two confidence levels $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ is non-negative on the entire support.

We next come to exponential spectral risk measures. First note that

$$R_{\phi_1}(p) = -\frac{\Phi_1''(p)}{\Phi_1'(p)} \geq R_{\phi_2}(p) = -\frac{\Phi_1''(p)}{\Phi_2'(p)} \text{ for all } p \in [0,1] \Rightarrow \Phi_1(p) \geq \Phi_2(p) \text{ for all } p \in [0,1] \tag{47}$$

(see the proof of Theorem 3.6 below). For exponential spectral risk measures,

$$R_{\phi_a}(p) = a, \tag{48}$$

which for $a_1 \geq a_2$ yields the assertion.

Likewise, for power spectral risk measures it holds that

$$R_{\phi_b}(p) = \frac{1-b}{p}, \tag{49}$$

which for $b_2 \geq b_1$ yields the assertion. $\qquad\square$

B. Proof of Theorem 3.6

If-part: We first show that $R_{\phi_1}(p) \geq R_{\phi_2}(p)$ for all $p \in [0,1] \Leftrightarrow \Phi_1(p) = g(\Phi_2(p))$ with $g(0) = 0$, $g(1) = 1$ and $g' > 0, g'' \leq 0$: It holds that

$$R_{\phi_1}(p) = -\frac{\Phi_1''(p)}{\Phi_1'(p)} = -\frac{\Phi_2''(p)}{\Phi_2'(p)} - \frac{g''(\Phi_2(p))}{g'(\Phi_2(p))} \cdot \Phi_2'(p) = R_{\phi_2}(p) - \frac{g''(\Phi_2(p))}{g'(\Phi_2(p))} \cdot \Phi_2'(p). \tag{50}$$

Hence, $R_{\phi_1}(p)$ is uniformly greater than $R_{\phi_2}(p)$ if and only if g satisfies $g' > 0, g'' \leq 0$.

Finally, for spectral risk measures we have $\Phi_1(p) = g(\Phi_2(p))$ with $g(0) = 0$, $g(1) = 1$ and $g' > 0, g'' \leq 0 \Rightarrow \Phi_1(p) - \Phi_2(p) \geq 0$ for all $p \in [0,1]$ due to the concavity of Φ. This proves the assertion.

Example 3.9 shows that the Only if-part does not hold. $\qquad\square$

C. Proof of Theorem 4.9

The difference $\Phi_{a_1}(p) - \Phi_{a_2}(p), a_1 > a_2$ is convex on $[\bar{p}, 1]$ if

$$\Phi_{a_1}''(p) - \Phi_{a_2}''(p) > 0 \tag{51}$$

$$\Leftrightarrow \frac{-a_1^2 \cdot e^{-a_1 \cdot p}}{1 - e^{-a_1}} > \frac{-a_2^2 \cdot e^{-a_2 \cdot p}}{1 - e^{-a_2}} \tag{52}$$

$$\Leftrightarrow p > \bar{p} = \frac{1}{a_2 - a_1} \cdot \ln\left(\frac{a_2^2 \cdot (1 - e^{-a_1})}{a_1^2 \cdot (1 - e^{-a_2})}\right) \tag{53}$$

It holds that $\bar{p} < 1$ if and only if

$$\frac{e^{a_1} - 1}{a_1^2} > \frac{e^{a_2} - 1}{a_2^2}. \tag{54}$$

The function

$$f(a) = \frac{e^a - 1}{a^2} \tag{55}$$

is decreasing on $(0, 1{,}5937]$ and increasing on $(1{,}5937, \infty)$, which proves 1. and 2., while 3. follows from (53) and (54). $\qquad\square$

D. Proof of Theorem 4.11

The difference $\Phi_{b_1}(p) - \Phi_{b_2}(p), 0 < b_1 < b_2 \leq 1$ is convex on $[\bar{p}, 1]$ if

$$\Phi_{b_1}''(p) - \Phi_{b_2}''(p) > 0 \tag{56}$$

$$\Leftrightarrow (b_1 - 1) \cdot b_1 \cdot p^{b_1} > (b_2 - 1) \cdot b_2 \cdot p^{b_2} \tag{57}$$

$$\Leftrightarrow p > \bar{p} = \left(\frac{b_2 \cdot (1 - b_2)}{b_1 \cdot (1 - b_1)}\right)^{\frac{1}{b_1 - b_2}} \tag{58}$$

It holds that $\bar{p} < 1$ if and only if

$$b_2 \cdot (1 - b_2) > b_1 \cdot (1 - b_1). \tag{59}$$

The function

$$f(b) = b \cdot (1 - b) \tag{60}$$

is increasing on $[0, 0{,}5]$ and decreasing on $(0{,}5, 1)$, which proves 1. and 2., while 3. follows from (58) and (59). $\qquad\square$

E. Proof of Theorem 4.15

The first order condition for the optimal risk free investment is given by

$$\frac{\partial \rho_\phi(\cdot)}{\partial z} = \frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) + (E(r) - r_f) \overset{!}{=} 0 \Leftrightarrow$$

$$\frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) \overset{!}{=} \rho_\phi(E(r) - r_f) \qquad (61)$$

First note that $\rho_\phi(X + (w_0 - z) \cdot r_0)$ is decreasing and convex in z for $z \in [0, w_0]$: $\rho_\phi(X + (w_0 - z) \cdot r_0)$ is decreasing in z, as decreasing z constitutes a mean preserving spread, which is rejected by any SRM-decision maker. $\rho_\phi(X + (w_0 - z) \cdot r_0)$ is convex in z, as the convexity of spectral risk measures (implied by subadditivity and positive homogeneity) for $z \in [0, w_0]$ yields

$$\rho_\phi(X + (w_0 - z) \cdot r_0) = \rho_\phi \left(\frac{z}{w_0} \cdot X + \frac{w_0 - z}{w_0} \cdot (X + w_0 \cdot r_0) \right)$$

$$\leq \frac{z}{w_0} \cdot \rho_\phi(X) + \frac{w_0 - z}{w_0} \cdot \rho_\phi(X + w_0 \cdot r_0). \qquad (62)$$

As $\rho_\phi(X + (w_0 - z) \cdot r_0)$ itself is decreasing and convex in z for $z \in [0, w_0]$, the left-hand side of (61), $\frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0)$, is negative and increasing in z for $z \in [0, w_0]$. The right-hand side of (61) is a negative constant. The optimal risk free investment is given at the intersection of the left-hand side and the right-hand side.

Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers. Then $z_{\phi_1}^*(X, r) \geq z_{\phi_2}^*(X, r)$ for all $X, r \in \mathcal{X}, r \neq E(r), X \neq E(X), E(r) > r_f, r_0 = r - E(r), E(r_0|X) = 0$ if and only if

$$\frac{\partial}{\partial z} \rho_{\phi_1}(X + (w_0 - z) \cdot r_0) \leq \frac{\partial}{\partial z} \rho_{\phi_2}(X + (w_0 - z) \cdot r_0) \text{ for all } z \in [0, w_0], \qquad (63)$$

i.e., if and only if the left-hand side of (61) for the SRM-decision maker with risk spectrum ϕ_1 is greater than (or equal to) the one of the SRM-decision maker with risk spectrum ϕ_2 for all $z \in [0, w_0]$. Condition (63) is satisfied if and only if

$$RP_{\phi_1}(X + (w_0 - z_2), (z_2 - z_1) \cdot r_0)$$

$$= \rho_{\phi_1}(X + (w_0 - z_1) \cdot r_0) - \rho_{\phi_1}(X + (w_0 - z_2) \cdot r_0)$$

$$\geq \rho_{\phi_2}(X + (w_0 - z_1) \cdot r_0) - \rho_{\phi_2}(X + (w_0 - z_2) \cdot r_0)$$

$$= RP_{\phi_2}(X + (w_0 - z_2), (z_{2-z_1}) \cdot r_0) \qquad (64)$$

for all $X, r \in \mathcal{X}, r \neq E(r), X \neq E(X), E(r) > r_f, r_0 = r - E(r), E(r_0|X) = 0$ and for all $0 \leq z_1 \leq z_2 \leq w_0$. This proves the assertion. $\qquad \square$

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